Eigenvalues of Schrödinger operators perturbed by dissipative barriers

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Introduction

Consider the Schrödinger operator

$$H_R := -\frac{d^2}{dx^2} + q + i\gamma \chi_{[0,R]} \quad \text{on} \quad L^2(\mathbb{R}_+) \quad (R > 0),$$

endowed with a Dirichlet boundary condition at 0, where:

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2. $\gamma > 0$.

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Spectral inclusion

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**Theorem 1** ([1])

(a) For any eigenvalue $\lambda$ of $H_\infty$, there exists eigenvalues $\lambda_R$ of $H_R$ and constants $C_0, \beta, R_0 > 0$ such that

$$|\lambda_R - \lambda| \leq C_0 e^{-\beta R} \quad (R \geq R_0).$$
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(b) If $\exists \varepsilon > 0 : \int e^{\varepsilon t} |q(t)| \, dt < \infty$, then for (almost$^1$) any $\mu \in \sigma_{\text{ess}}(H_\infty)$ there exists eigenvalues $\lambda_R$ of $H_R$ and constants $C_0, R_0 > 0$ such that

$$|\lambda_R - \mu| \leq \frac{C_0}{R} \quad (R \geq R_0).$$

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Example

\( R = 50, q = i\chi_{[0, 4.7]}, \gamma = 1 \)
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- Enclosures for the eigenvalues of \( H_R \).
- Estimates for the number of eigenvalues \( N(H_R) \).
**Enclosures**

**Theorem 2 ([2])**

(a) \( \exists \) constant \( X = X(q, \gamma) > 0 \) such that \( \forall R > 0 \) the eigenvalues of \( H_R \) lie in

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\Gamma := B_X(0) \cup ([0, \infty) + i[0, \gamma])
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To compare to (b), application of a sharp enclosure of Frank, Laptev and Seiringer (2011) gives \( \sqrt{|\lambda_R|} = O(R). \)
Ideas in proof of Theorem 2.

1. **Use large-$|\lambda|$ Levinson asymptotics:** Solutions $\psi_{\pm}(\cdot, \lambda)$ to $-\psi'' + q\psi = \lambda \psi$ such that

$$\psi_{\pm}(x, \lambda) = e^{\pm i \sqrt{\lambda} x} (1 + E_{\pm}(x, \lambda))$$

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2. Construct analytic function \( f_R \) such that

\[
\lambda \text{ eigenvalue of } H_R \iff f_R(\lambda) = 0.
\]

\( f_R \) has form

\[
f_R(\lambda) = \psi_-(0, \lambda - i\gamma) \left( \sqrt{\lambda} - \sqrt{\lambda - i\gamma} + \mathcal{E}_1(R, \lambda) \right) e^{i \sqrt{\lambda - i\gamma} R}
\]

\[
- \psi_+(0, \lambda - i\gamma) \left( \sqrt{\lambda} + \sqrt{\lambda - i\gamma} + \mathcal{E}_2(R, \lambda) \right) e^{-i \sqrt{\lambda - i\gamma} R}
\]

where

\[
|\mathcal{E}_1(R, \lambda)| + |\mathcal{E}_2(R, \lambda)| \leq C(q, \gamma).
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Counting eigenvalues

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N(H_R) \leq C \frac{\sqrt{X} + a}{a^2} \frac{\gamma R^3}{(\log R)^2} \quad (R \geq R_0)
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where \( X = X(q, \gamma) > 0 \) appeared in Th. 2 (a) and \( C = 88788 \).
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   ▶ Apply Jensen’s formula to bound $N(H_R)$ and prove Th. 3 (a).
Ideas in proof of Theorem 3. (cont.)

3. \( \exists a > 0 : \int e^{4at} |q(t)| \, dt < \infty \)
   \[ \Rightarrow g_R \text{ admits analytic continuation to } \{ \Im z > -2a \} \]
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- Prove bound for number of zeros of an arbitrary analytic function on \( \{ \Im z > -2a \} \) in the region

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- Apply this bound to \( g_R \) with \( r = O(R/\log R) \).
Thanks for listening!

References


Appendix
Proposition

Suppose that $f$ is an analytic function defined on an open neighbourhood of the closed semi-disc $D_r := \overline{B}_r(0) \cap \mathbb{C}_+$ for some $r > 0$. Let $\alpha$ and $\beta$ be any numbers in the interval $(0, 1)$ satisfying

$$\beta \left( \frac{1 - \alpha}{\alpha + \beta} \right)^2 > \frac{Y}{\eta} \quad (1)$$

and let $N(\alpha r)$ denote the number of zeros in the region

$$D_{\alpha r, \eta, Y} := \{ z \in \mathbb{C} : \eta \leq \Im z \leq Y, |z| \leq \alpha r \} \quad (2)$$

where $Y, \eta > 0$ are given parameters satisfying $\eta < Y < r$. Then,

$$N(\alpha r) \leq \frac{2}{\log \Lambda(r)} \log \left( \frac{1}{\min\{\beta, 1 - \beta\}} \frac{\sup_{z \in \partial D_r} |f(z)|}{|f(i \beta r)|} \right) \quad (3)$$

where

$$\Lambda(r) := \frac{1 + \frac{4\beta \eta}{(\alpha + \beta)^2} \frac{1}{r}}{1 + \frac{4Y}{(1-\alpha)^2}\frac{1}{r}} \quad (4)$$

Remark

One can always guarantee that condition (1) for $\alpha$ and $\beta$ is satisfied by choosing, for instance,

$$\alpha = \beta = \frac{1}{4} \frac{\eta}{2Y + \eta} \quad (5)$$